

# On a problem of Janusz Matkowski and Jacek Wesółowski

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**Abstract.** We study the problem of the existence of increasing and continuous solutions  $\varphi: [0, 1] \rightarrow [0, 1]$  such that  $\varphi(0) = 0$  and  $\varphi(1) = 1$  of the functional equation

$$\varphi(x) = \sum_{n=0}^N \varphi(f_n(x)) - \sum_{n=1}^N \varphi(f_n(0)),$$

where  $N \in \mathbb{N}$  and  $f_0, \dots, f_N: [0, 1] \rightarrow [0, 1]$  are strictly increasing contractions satisfying the following condition  $0 = f_0(0) < f_0(1) = f_1(0) < \dots < f_{N-1}(1) = f_N(0) < f_N(1) = 1$ . In particular, we give an answer to the problem posed in [9] by Janusz Matkowski concerning a very special case of that equation.

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## 1. Introduction

During the 47th International Symposium on Functional Equations in 2009 Jacek Wesółowski asked whether the identity on  $[0, 1]$  is the only increasing and continuous solution  $\varphi: [0, 1] \rightarrow [0, 1]$  of the equation

$$\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right) \quad (\text{e}_1)$$

satisfying

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1. \quad (1)$$

This question has been posed in connection with studying probability measures in the plane which are invariant by “winding” (see [10]).

A negative answer to this question has been obtained in [5] and reads as follows.

- Theorem 1.1.** (i) *The identity on  $[0, 1]$  is the only increasing and absolutely continuous solution  $\varphi: [0, 1] \rightarrow [0, 1]$  of equation  $(e_1)$  satisfying (1).*  
(ii) *For every  $p \in (0, 1)$  the function  $\varphi_p: [0, 1] \rightarrow [0, 1]$  given by*

$$\varphi_p \left( \sum_{k=1}^{\infty} \frac{x_k}{2^k} \right) = \sum_{k=1}^{\infty} x_k p^{k - \sum_{i=1}^{k-1} x_i} (1-p)^{\sum_{i=1}^{k-1} x_i}, \quad (2)$$

where  $x_k \in \{0, 1\}$  for all  $k \in \mathbb{N}$ , is an increasing and continuous solution of equation  $(e_1)$  satisfying (1). Moreover,  $\varphi_p$  is singular for every  $p \neq \frac{1}{2}$ .

Let us note that the first assertion of Theorem 1.1 is known (see e.g. [13] or [8]), however in [5] we can find an independent proof of it.

It turns out that in 1985 Janusz Matkowski posed a problem asking if equation  $(e_1)$  has a non-linear monotonic and continuous solution  $\varphi: [0, 1] \rightarrow \mathbb{R}$  (see [9]). Moreover, he observed that monotonic solutions of equation  $(e_1)$  are connected with invariant measures for a certain map on  $[0, 1]$ . Note that Matkowski's problem is equivalent to Wesółowski's question.

- Remark 1.2.* (i) If  $\varphi: [0, 1] \rightarrow [0, 1]$  is an increasing and continuous solution of equation  $(e_1)$  satisfying (1), then for all  $a, b \in \mathbb{R}$  the function  $a\varphi + b$  is monotonic, continuous and satisfies  $(e_1)$  for every  $x \in [0, 1]$ .  
(ii) If  $\varphi: [0, 1] \rightarrow \mathbb{R}$  is a monotonic and continuous solution of equation  $(e_1)$ , different from a constant function, then  $\frac{\varphi - \varphi(0)}{\varphi(1) - \varphi(0)}$  is an increasing and continuous function satisfying (1) and  $(e_1)$  for every  $x \in [0, 1]$ .

## 2. Preliminaries

Fix  $N \in \mathbb{N}$ , strictly increasing contractions  $f_0, \dots, f_N: [0, 1] \rightarrow [0, 1]$  such that

$$0 = f_0(0) < f_0(1) = f_1(0) < \dots < f_{N-1}(1) = f_N(0) < f_N(1) = 1 \quad (3)$$

and consider the functional equation

$$\varphi(x) = \sum_{n=0}^N \varphi(f_n(x)) - \sum_{n=1}^N \varphi(f_n(0)) \quad (E)$$

for every  $x \in [0, 1]$ . Denote by  $\mathcal{C}$  the class of all continuous and increasing solutions  $\varphi: [0, 1] \rightarrow [0, 1]$  of equation (E) satisfying (1). Following the idea from [5] we show that  $\mathcal{C}$  contains many functions, however, we manage to identify a quite large class of contractions that includes the similitudes such that there is exactly one absolutely continuous solution.

We begin with two observations showing that in many situations the class  $\mathcal{C}$  is determined by two of its subclasses  $\mathcal{C}_a$  and  $\mathcal{C}_s$ , consisting of all absolutely continuous and all singular functions, respectively.

*Remark 2.1.* If  $\varphi_1, \varphi_2 \in \mathcal{C}$  and if  $\alpha \in (0, 1)$ , then  $\alpha\varphi_1 + (1 - \alpha)\varphi_2 \in \mathcal{C}$ .

To formulate the next remark we recall that a Lebesgue measurable function  $f: [0, 1] \rightarrow [0, 1]$  is said to be *nonsingular* if the set  $f^{-1}(A)$  has Lebesgue measure zero for every set  $A \subset [0, 1]$  of Lebesgue measure zero (see [6]). Observe that an invertible Lebesgue measurable function  $f$  is nonsingular if and only if its inverse  $f^{-1}$  satisfies Luzin's condition (N).

*Remark 2.2.* Assume that all the contractions  $f_0, \dots, f_N$  are nonsingular. Then, both the absolutely continuous and the singular parts of every element from  $\mathcal{C}$  satisfy (E) for every  $x \in [0, 1]$ .

*Proof.* Fix  $\varphi \in \mathcal{C}$  and denote by  $\varphi_a$  and  $\varphi_s$  its absolutely continuous and singular parts<sup>1</sup>, respectively. By (E), for every  $x \in [0, 1]$  we have

$$\varphi_a(x) - \sum_{n=0}^N \varphi_a(f_n(x)) = -\varphi_s(x) + \sum_{n=0}^N \varphi_s(f_n(x)) - \sum_{n=1}^N \varphi(f_n(0)),$$

and hence there exists a real constant  $c$  such that

$$\varphi_a(x) - \sum_{n=0}^N \varphi_a(f_n(x)) = c \quad \text{and} \quad -\varphi_s(x) + \sum_{n=0}^N \varphi_s(f_n(x)) - \sum_{n=1}^N \varphi(f_n(0)) = c.$$

This jointly with the fact  $f_0(0) = 0$  stipulated in (3) gives

$$c = \varphi_a(0) - \sum_{n=0}^N \varphi_a(f_n(0)) = - \sum_{n=1}^N \varphi_a(f_n(0)),$$

and in consequence

$$\varphi_a(x) = \sum_{n=0}^N \varphi_a(f_n(x)) - \sum_{n=1}^N \varphi_a(f_n(0))$$

and

$$\varphi_s(x) = \sum_{n=0}^N \varphi_s(f_n(x)) - \sum_{n=1}^N \varphi_s(f_n(0))$$

for every  $x \in [0, 1]$ . □

For all  $k \in \mathbb{N}$  and  $n_1, \dots, n_k \in \{0, \dots, N\}$  denote by  $f_{n_1, \dots, n_k}$  the composition  $f_{n_1} \circ \dots \circ f_{n_k}$ . We extend the notation to the case  $k = 0$  by letting  $f_{n_1, \dots, n_0}$  being the identity.

**Lemma 2.3.** *Let  $(n_k)_{k \in \mathbb{N}}$  be a sequence of elements of  $\{0, \dots, N\}$ . Then the sequence  $(f_{n_1, \dots, n_k}(0))_{k \in \mathbb{N}}$  is increasing and the sequence  $(f_{n_1, \dots, n_k}(1))_{k \in \mathbb{N}}$  is decreasing. Moreover,*

$$\lim_{k \rightarrow \infty} f_{n_1, \dots, n_k}(y) = \lim_{k \rightarrow \infty} f_{n_1, \dots, n_k}(z)$$

for all  $y, z \in [0, 1]$ .

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<sup>1</sup>The parts are unique up to a constant. For definiteness, we choose them such that  $\varphi_a(0) = \varphi_s(0) = 0$ .

*Proof.* Fix a sequence  $(n_k)_{k \in \mathbb{N}}$  of elements of  $\{0, \dots, N\}$  and an integer number  $k \geq 2$ . From (3) we have

$$0 \leq f_{n_k}(0) < f_{n_k}(1) \leq 1$$

and by the strict monotonicity of  $f_{n_1, \dots, n_{k-1}}$  we conclude that

$$f_{n_1, \dots, n_{k-1}}(0) \leq f_{n_1, \dots, n_k}(0) < f_{n_1, \dots, n_k}(1) \leq f_{n_1, \dots, n_{k-1}}(1).$$

To complete the proof it is enough to observe that for all  $y, z \in [0, 1]$  and  $k \in \mathbb{N}$  we have

$$|f_{n_1, \dots, n_k}(y) - f_{n_1, \dots, n_k}(z)| \leq f_{n_1, \dots, n_k}(1) - f_{n_1, \dots, n_k}(0) \leq c^k,$$

where  $c \in (0, 1)$  is the largest Lipschitz constant of the given contractions  $f_0, \dots, f_N$ .  $\square$

**Lemma 2.4.** *For every  $x \in [0, 1]$  there exists a sequence  $(x_k)_{k \in \mathbb{N}}$  of elements of  $\{0, \dots, N\}$  such that*

$$x = \lim_{k \rightarrow \infty} f_{x_1, \dots, x_k}(0). \quad (4)$$

*Proof.* Fix  $x \in [0, 1]$  and observe that according to Lemma 2.3 it is enough to show that there exists a sequence  $(x_k)_{k \in \mathbb{N}}$  of elements of  $\{0, \dots, N\}$  such that

$$f_{x_1, \dots, x_k}(0) \leq x \leq f_{x_1, \dots, x_k}(1) \quad (5)$$

for every  $k \in \mathbb{N}$ .

By (3) there exists  $x_1 \in \{0, \dots, N\}$  such that

$$f_{x_1}(0) \leq x \leq f_{x_1}(1).$$

Thus, (5) holds for  $k = 1$ .

Fix  $k \in \mathbb{N}$  and assume inductively that there exist  $x_1, \dots, x_k \in \{0, \dots, N\}$  such that (5) holds. Then

$$0 \leq f_{x_1, \dots, x_k}^{-1}(x) \leq 1$$

and by (3) there exists  $x_{k+1} \in \{0, \dots, N\}$  such that

$$f_{x_{k+1}}(0) \leq f_{x_1, \dots, x_k}^{-1}(x) \leq f_{x_{k+1}}(1).$$

Hence

$$f_{x_1, \dots, x_{k+1}}(0) \leq x \leq f_{x_1, \dots, x_{k+1}}(1),$$

and the proof is complete.  $\square$

### 3. General case

Fix positive real numbers  $p_0, \dots, p_N$  such that

$$\sum_{n=0}^N p_n = 1. \quad (6)$$

Then there exists a unique Borel probability measure  $\mu$  such that

$$\mu(A) = \sum_{n=0}^N p_n \mu(f_n^{-1}(A)) \quad (7)$$

for every Borel set  $A \subset [0, 1]$  (see [4]; cf. [3]). From now on the letter  $\mu$  will be reserved for the unique Borel probability measure satisfying (7) for every Borel set  $A \subset [0, 1]$ .

**Lemma 3.1.** *The measure  $\mu$  is continuous.*

*Proof.* As a first step we want to show that

$$\mu(\{f_n(0)\}) = \mu(\{f_n(1)\}) = 0 \quad (8)$$

for every  $n \in \{0, \dots, N\}$ .

Applying (7) and using (3), we obtain

$$\mu(\{0\}) = \mu(\{f_0(0)\}) = \sum_{n=0}^N p_n \mu(\{f_n^{-1}(f_0(0))\}) = p_0 \mu(\{0\}) + \sum_{n=1}^N p_n \mu(\emptyset),$$

By the fact that  $p_0 \in (0, 1)$  we conclude that

$$\mu(\{f_0(0)\}) = \mu(\{0\}) = 0.$$

Similarly, applying (7), (3) and the fact that  $p_N \in (0, 1)$  we conclude that

$$\mu(\{f_N(1)\}) = \mu(\{1\}) = 0.$$

If  $n \in \{1, \dots, N\}$ , then applying again (7) and (3), we obtain

$$\mu(\{f_{n-1}(1)\}) = \mu(\{f_n(0)\}) = p_{n-1} \mu(\{1\}) + p_n \mu(\{0\}) = 0.$$

Our second step is to prove that

$$\mu([f_{n_1, \dots, n_k}(0), f_{n_1, \dots, n_k}(1)]) = \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k\} : n_i = n\}} \quad (9)$$

for all  $k \in \mathbb{N} \cup \{0\}$  and  $n_1, \dots, n_k \in \{0, \dots, N\}$ .

Since  $\mu([0, 1]) = 1$ , it follows that (9) is satisfied for  $k = 0$ .

Fix  $k \in \mathbb{N} \cup \{0\}$  and assume that (9) holds for all  $n_1, \dots, n_k \in \{0, \dots, N\}$ .

Fix also  $n_{k+1} \in \{0, \dots, N\}$ .

Note first that from (8), (3), and (7), we get

$$\mu(B) = p_n \mu(f_n^{-1}(B)) \quad (10)$$

for all  $n \in \{0, \dots, N\}$  and Borel sets  $B \subset [f_n(0), f_n(1)]$ . This jointly with (9) implies

$$\begin{aligned} \mu([f_{n_1, \dots, n_{k+1}}(0), f_{n_1, \dots, n_{k+1}}(1)]) &= p_{n_1} \mu([f_{n_2, \dots, n_{k+1}}(0), f_{n_2, \dots, n_{k+1}}(1)]) \\ &= p_{n_1} \prod_{n=0}^N p_n^{\#\{i \in \{2, \dots, k+1\} : n_i = n\}} \\ &= \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k+1\} : n_i = n\}}. \end{aligned}$$

To prove that  $\mu$  is continuous it is sufficient to show that  $\mu$  has no atoms.

Fix  $x \in [0, 1]$ . From Lemma 2.4 we conclude that there exists a sequence  $(x_k)_{k \in \mathbb{N}}$  of elements of  $\{0, \dots, N\}$  such that (4) holds. Then applying Lemma 2.3 and (9) with  $n_i = x_i$  for  $i \in \{1, \dots, k\}$ , we obtain

$$\begin{aligned} \mu(\{x\}) &= \mu \left( \bigcap_{k \in \mathbb{N}} [f_{x_1, \dots, x_k}(0), f_{x_1, \dots, x_k}(1)] \right) \\ &= \lim_{k \rightarrow \infty} \mu([f_{x_1, \dots, x_k}(0), f_{x_1, \dots, x_k}(1)]) \\ &= \lim_{k \rightarrow \infty} \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k\} : x_i = n\}} \leq \lim_{k \rightarrow \infty} (\max\{p_0, \dots, p_N\})^k = 0, \end{aligned}$$

and the proof is complete.  $\square$

The next lemma is folklore (the reader can consult [2, 12] in the case where  $f_0, \dots, f_N$  are similitudes and [7] in the case where  $f_0, \dots, f_N$  are contractions). More general results in this direction can be found e.g. in [14, 15].

**Lemma 3.2.** *The measure  $\mu$  is either singular or absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$ .*

Define the function  $\varphi: [0, 1] \rightarrow [0, 1]$  by

$$\varphi(x) = \mu([0, x]).$$

From now on the letter  $\varphi$  will be reserved for the just defined function.

**Theorem 3.3.** *Either  $\varphi \in \mathcal{C}_a$  or  $\varphi \in \mathcal{C}_s$ .*

*Proof.* We first prove that  $\varphi \in \mathcal{C}$ .

That  $\varphi$  is increasing is a consequence of the monotonicity of  $\mu$ . The continuity of  $\varphi$  and that  $\varphi(0) = 0$  follows from Lemma 3.1. Since  $\mu$  is a probability measure, we have  $\varphi(1) = 1$ .

From (10) we get

$$\mu(f_n(B)) = p_n \mu(B)$$

for all  $n \in \{0, \dots, N\}$  and Borel sets  $B \subset [0, 1]$ . This jointly with (6) gives

$$\sum_{n=0}^N \mu(f_n(B)) = \sum_{n=0}^N p_n \mu(B) = \mu(B)$$

for every Borel set  $B \subset [0, 1]$ . Hence,

$$\begin{aligned} \varphi(x) &= \mu([0, x]) = \sum_{n=0}^N \mu(f_n([0, x])) = \sum_{n=0}^N \mu([f_n(0), f_n(x)]) \\ &= \sum_{n=0}^N \mu([0, f_n(x)]) - \sum_{n=0}^N \mu([0, f_n(0)]) \\ &= \sum_{n=0}^N \varphi(f_n(x)) - \sum_{n=0}^N \varphi(f_n(0)) = \sum_{n=0}^N \varphi(f_n(x)) - \sum_{n=1}^N \varphi(f_n(0)) \end{aligned}$$

for every  $x \in [0, 1]$ .

Thus, we have proved that  $\varphi \in \mathcal{C}$ . Now the assertion of the lemma follows from Lemma 3.2; to see it the reader can consult [1, Theorem 31.7].  $\square$

It is a very difficult (and still open) problem to decide for which parameters  $p_0, \dots, p_N$  the function  $\varphi$  is absolutely continuous. However, it turns out that under some assumptions on the given contractions  $f_0, \dots, f_N$  equation (E) has exactly one absolutely continuous solution in the class  $\mathcal{C}$ .

**Theorem 3.4.** *Assume that  $f_0, \dots, f_N \in C^2([0, 1])$  and there exist  $\lambda \in (0, 1)$  and  $c \in (0, \infty)$  such that  $0 < f'_n(x) \leq \lambda$  and  $f''_n(x) \leq cf'_n(x)$  for all  $n \in \{0, \dots, N\}$  and  $x \in [0, 1]$ . Then  $\mathcal{C}_a$  consists of exactly one function.*

*Proof.* Define  $S: [0, 1] \rightarrow [0, 1]$  by

$$S(x) = \begin{cases} f_n^{-1}(x) & \text{for } x \in [f_n(0), f_n(1)) \text{ and } n \in \{0, \dots, N\}, \\ 1 & \text{for } x = 1. \end{cases}$$

Now it is enough to apply [6, Theorem 6.2.1].  $\square$

Theorem 3.4 enforces looking for these unique parameters  $p_0, \dots, p_N$  for which  $\varphi \in \mathcal{C}_a$ . It is still difficult in full generality. However, it can be done with success in the case where  $f_0, \dots, f_N$  are similitudes; such a case will be considered in the next section.

Now let us set down an obvious characterization of these contractions  $f_0, \dots, f_N$  for which  $\text{id}_{[0,1]} \in \mathcal{C}_a$ .

**Proposition 3.5.** *The identity on  $[0, 1]$  belongs to  $\mathcal{C}_a$  if and only if*

$$\sum_{n=0}^N f_n(x) - x = \sum_{n=1}^N f_n(0) \quad (11)$$

for every  $x \in [0, 1]$ .

The last result of this section gives a precise formula for  $\varphi$ .

**Theorem 3.6.** *Assume that  $x \in [0, 1]$  and let  $(x_k)_{k \in \mathbb{N}}$  be a sequence of elements of  $\{0, \dots, N\}$  such that (4) holds. Then*

$$\varphi(x) = \sum_{k=1}^{\infty} \text{sgn}(x_k) \left[ \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k-1\} : x_i = n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right].$$

*Proof.* We begin with showing inductively that

$$\mu([f_{n_1, \dots, n_{k-1}}(0), f_{n_1, \dots, n_k}(0)]) = \text{sgn}(n_k) \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k-1\} : n_i = n\}} \cdot \sum_{n=0}^{n_k-1} p_n \quad (12)$$

for all  $k \in \mathbb{N}$  all  $n_1, \dots, n_k \in \{0, \dots, N\}$ .

If  $n_1 = 0$ , then  $\text{sgn}(n_1) = 0$ , and hence

$$\mu([0, f_{n_1}(0)]) = \mu(\{0\}) = 0 = \text{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

If  $n_1 \geq 1$ , we have  $\text{sgn}(n_1) = 1$ , and then by (3), (10) and Lemma 3.1 we obtain

$$\mu([0, f_{n_1}(0)]) = \sum_{n=0}^{n_1-1} \mu([f_n(0), f_n(1)]) = \sum_{n=0}^{n_1-1} p_n \mu([0, 1]) = \text{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

Therefore (12) holds for  $k = 1$  and all  $n_1, \dots, n_k \in \{0, \dots, N\}$ .

Fix  $k \in \mathbb{N}$  and assume that (12) holds for all  $n_1, \dots, n_k \in \{0, \dots, N\}$ .

Fix  $n_{k+1} \in \{0, \dots, N\}$ . Applying (10) and (12) we get

$$\begin{aligned} \mu([f_{n_1, \dots, n_k}(0), f_{n_1, \dots, n_{k+1}}(0)]) &= p_{n_1} \mu([f_{n_2, \dots, n_k}(0), f_{n_2, \dots, n_{k+1}}(0)]) \\ &= p_{n_1} \text{sgn}(n_{k+1}) \prod_{n=0}^N p_n^{\#\{i \in \{2, \dots, k\} : n_i = n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n \\ &= \text{sgn}(n_{k+1}) \prod_{n=0}^N p_n^{\#\{i \in \{1, 2, \dots, k\} : n_i = n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n. \end{aligned}$$

By the continuity of  $\varphi$  (see Theorem 3.3) we have

$$\varphi(x) = \varphi\left(\lim_{l \rightarrow \infty} f_{x_1, \dots, x_l}(0)\right) = \lim_{l \rightarrow \infty} \varphi(f_{x_1, \dots, x_l}(0)).$$

Then using (3), Lemma 3.1 and (12) with  $n_i = x_i$  for all  $i \in \{1, \dots, l\}$ , we get

$$\begin{aligned} \varphi(f_{x_1, \dots, x_l}(0)) &= \mu([0, f_{x_1, \dots, x_l}(0)]) = \sum_{k=1}^l \mu([f_{x_1, \dots, x_{k-1}}(0), f_{x_1, \dots, x_k}(0)]) \\ &= \sum_{k=1}^l \text{sgn}(x_k) \left[ \prod_{n=0}^N p_n^{\#\{i \in \{1, \dots, k-1\} : x_i = n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right]. \end{aligned}$$

Passing with  $l$  to  $\infty$  we obtain the required formula for  $\varphi$ .  $\square$

#### 4. Similitudes case

Throughout this section we assume that  $f_0, \dots, f_N$  are similitudes, i.e. there exist real numbers  $\rho_0, \dots, \rho_N \in (0, 1)$  such that

$$\sum_{n=0}^N \rho_n = 1 \tag{13}$$

and

$$f_n(x) = \rho_n x + \sum_{k=0}^{n-1} \rho_k$$

for all  $x \in [0, 1]$  and  $n \in \{0, \dots, N\}$ .

Note that (3) holds.

Since the above defined similitudes satisfy the assumptions of Theorem 3.4, it follows that the class  $\mathcal{C}$  has exactly one absolutely continuous



solution. Thus according to Theorem 3.3 we conclude that  $\varphi$  is singular except one very particular case of parameters  $p_0, \dots, p_N$ , which we are looking for.

**Theorem 4.1.** *If  $p_n = \rho_n$  for every  $n \in \{0, \dots, N\}$ , then  $\varphi = \text{id}_{[0,1]}$ .*

*Proof.* Assume that  $p_n = \rho_n$  for every  $n \in \{0, \dots, N\}$ .

Observe first that applying (13), we get

$$\sum_{n=0}^N f_n(x) - x = \sum_{n=0}^N \rho_n x + \sum_{n=0}^N \sum_{k=0}^{n-1} \rho_k - x = \sum_{n=0}^N f_n(0) = \sum_{n=1}^N f_n(0)$$

for every  $x \in [0, 1]$ . Thus,  $\text{id}_{[0,1]} \in \mathcal{C}_a$ , by Proposition 3.5.

Now we can use Theorem 3.4 or argue as follows.

Denote by  $\nu$  the one-dimensional Lebesgue measure restricted to  $[0, 1]$ . According to [1, Theorem 12.4] we infer that  $\nu$  is the unique Borel measure on  $[0, 1]$  such that  $\nu([0, x]) = x$  for every  $x \in [0, 1]$ . Fix  $n \in \{0, \dots, N\}$  and choose  $x \in [f_n(0), f_n(1)]$ . Then

$$\begin{aligned} \nu([f_n(0), x]) &= \nu([0, x]) - \nu([0, f_n(0)]) = x - f_n(0) = \rho_n \left( \frac{x}{\rho_n} - \sum_{k=0}^{n-1} \frac{\rho_k}{\rho_n} \right) \\ &= p_n f_n^{-1}(x) = p_n \nu([0, f_n^{-1}(x)]) = p_n \nu(f_n^{-1}([f_n(0), x])). \end{aligned}$$

Hence

$$\nu(A) = p_n \nu(f_n^{-1}(A))$$

for every Borel set  $A \subset [f_n(0), f_n(1)]$ , and in consequence,

$$\nu(A) = \sum_{n=0}^N p_n \nu(f_n^{-1}(A))$$

for every Borel set  $A \subset [0, 1]$ . Finally, by the uniqueness of  $\mu$  we obtain

$$\varphi(x) = \mu([0, x]) = \nu([0, x]) = x$$

for every  $x \in [0, 1]$ . □

Combining Theorems 3.3, 3.4 and 4.1 we get the following corollary.

**Corollary 4.2.** *If  $p_n \neq \rho_n$  for some  $n \in \{0, \dots, N\}$ , then  $\varphi \in \mathcal{C}_s$ .*

Note that in our setting  $\prod_{n=0}^N p_n^{p_n} \rho_n^{-p_n} \geq 1$ . Observe also that the iterated function system consisting of the contractions  $f_0, \dots, f_N$  satisfies the open set condition. Therefore Theorem 4.1 jointly with Corollary 4.2 can be written in the following form, which corresponds to Theorem 1.1 from [11].

**Theorem 4.3.** *We have  $\varphi \in \mathcal{C}_a$  if and only if  $p_n = \rho_n$  for every  $n \in \{0, \dots, N\}$ . Moreover, if  $\varphi \in \mathcal{C}_a$ , then  $\varphi = \text{id}_{[0,1]}$ .*

To the end of this section we assume that

$$\rho_0 = \rho_1 = \dots = \rho_N = \frac{1}{N+1}.$$

Note that (13) is satisfied and equation (E) now takes the form

$$\varphi(x) = \sum_{n=0}^N \varphi\left(\frac{x+n}{N+1}\right) - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right). \quad (\mathbf{e}_N)$$

It is clear that for  $N = 1$  equation  $(\mathbf{e}_N)$  reduces to equation  $(\mathbf{e}_1)$ .

Fix  $x \in [0, 1]$  and define a sequence  $(x_k)_{k \in \mathbb{N}}$  of elements of  $\{0, \dots, N\}$  as follows:

if  $x = 1$  we put  $x_k = N$  for every  $k \in \mathbb{N}$ ;

if  $x < 1$  we put  $x_1 = [(N+1)x]$  and then inductively

$$x_{k+1} = \left[ (N+1)^{k+1}x - \sum_{i=1}^k (N+1)^{k+1-i}x_i \right]$$

for every  $k \in \mathbb{N}$ , where  $[y]$  denotes the integer part of  $y \in \mathbb{R}$ .

Clearly,

$$x = \lim_{k \rightarrow \infty} f_{x_1, \dots, x_k}(0) = \sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k},$$

and Theorem 3.6 yields

$$\varphi\left(\sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k}\right) = \sum_{k=1}^{\infty} \operatorname{sgn}(x_k) \left[ \prod_{n=0}^N p_n^{\#\{i \in \{1, 2, \dots, k-1\} : x_i = n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right]. \quad (14)$$

In particular,

$$\varphi\left(\frac{n}{N+1}\right) = \sum_{k=0}^{n-1} p_k \quad (15)$$

for every  $n \in \{1, \dots, N\}$ .

Now we are able to calculate the integral of  $\varphi$  on  $[0, 1]$ .

**Proposition 4.4.** *We have*

$$\int_0^1 \varphi(x) dx = \frac{1}{N} \sum_{n=1}^N n p_{N-n}.$$

*Proof.* Using (15) and  $(\mathbf{e}_N)$ , we get

$$\begin{aligned} \int_0^1 \varphi(x) dx &= \sum_{n=0}^N \int_0^1 \varphi\left(\frac{x+n}{N+1}\right) dx - \int_0^1 \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) dx \\ &= (N+1) \sum_{n=0}^N \int_{\frac{n}{N+1}}^{\frac{n+1}{N+1}} \varphi(y) dy - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= (N+1) \int_0^1 \varphi(x) dx - \sum_{k=0}^{N-1} (N-k) p_k. \end{aligned}$$

This implies the required formula for the integral of  $\varphi$ .  $\square$

We end this section observing that  $\varphi$  can be extended to an increasing and continuous function satisfying  $(\mathbf{e}_N)$  for every  $x \in \mathbb{R}$ .

**Proposition 4.5.** *The function  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  given by*

$$\phi(x) = [x] + \varphi(x - [x])$$

*is increasing, continuous and satisfies  $(e_N)$  for every  $x \in \mathbb{R}$ .*

*Proof.* Fix  $x \in \mathbb{R}$  and assume that  $x \in [m(N+1)+l, m(N+1)+l+1)$  for some  $m \in \mathbb{Z}$  and  $l \in \{0, 1, \dots, N\}$ . Then  $[\frac{x+i}{N+1}] = m$  for every  $i \in \{0, \dots, N-l\}$  and  $[\frac{x+i}{N+1}] = m+1$  for every  $i \in \{N-l+1, \dots, N\}$ . Consequently,

$$\begin{aligned} \phi(x) &= [x] + \varphi(x - [x]) = \\ &= m(N+1) + l + \sum_{n=0}^N \varphi\left(\frac{x - [x] + n}{N+1}\right) - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= m(N+1) + l + \sum_{n=0}^N \varphi\left(\frac{x + n - l}{N+1} - m\right) - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= (m+1)l + \sum_{n=0}^{l-1} \varphi\left(\frac{x + n - l + N + 1}{N+1} - m - 1\right) \\ &\quad + m(N+1-l) + \sum_{n=l}^N \varphi\left(\frac{x + n - l}{N+1} - m\right) - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= (m+1)l + \sum_{n=N+1-l}^N \varphi\left(\frac{x + n}{N+1} - m - 1\right) \\ &\quad + m(N-l+1) + \sum_{n=0}^{N-l} \varphi\left(\frac{x + n}{N+1} - m\right) - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= \sum_{n=N-l+1}^N \left\{ \left[ \frac{x+n}{N+1} \right] + \varphi\left(\frac{x+n}{N+1} - \left[ \frac{x+n}{N+1} \right] \right) \right\} \\ &\quad + \sum_{n=0}^{N-l} \left\{ \left[ \frac{x+n}{N+1} \right] + \varphi\left(\frac{x+n}{N+1} - \left[ \frac{x+n}{N+1} \right] \right) \right\} - \sum_{n=1}^N \varphi\left(\frac{n}{N+1}\right) \\ &= \sum_{n=0}^N \phi\left(\frac{x+n}{N+1}\right) - \sum_{n=1}^N \phi\left(\frac{n}{N+1}\right). \end{aligned}$$

To prove that  $\phi$  is increasing fix  $x < y$ . If  $[x] = [y]$ , then

$$\phi(x) = [x] + \varphi(x - [x]) = [y] + \varphi(x - [y]) \leq [y] + \varphi(y - [y]) = \phi(y),$$

and if  $[x] < [y]$ , then

$$\phi(x) = [x] + \varphi(x - [x]) \leq [y] \leq [y] + \varphi(y - [y]) = \phi(y).$$

It is clear that  $\phi$  is continuous at every point of the set  $\mathbb{R} \setminus \mathbb{Z}$ . If  $k \in \mathbb{Z}$ , then by the continuity of  $\varphi$  and (1) we obtain

$$\lim_{x \rightarrow k^+} \phi(x) = \lim_{x \rightarrow k^+} ([x] + \varphi(x - [x])) = k + \lim_{y \rightarrow 0^+} \varphi(y) = k = \phi(k)$$

and

$$\lim_{x \rightarrow k^-} \phi(x) = \lim_{x \rightarrow k^-} ([x] + \varphi(x - [x])) = k - 1 + \lim_{y \rightarrow 1^-} \varphi(y) = k,$$

which completes the proof.  $\square$

## 5. Matkowski-Wesołowski case

First of all observe that formula (14) with  $N = 1$  coincides with formula (2). So the main part of assertion (ii) of Theorem 1.1 is a very special case of Theorem 3.6, whereas its moreover part follows from Corollary 4.2. Now we would like to get a little bit more information about the class  $\mathcal{C}$ . For this purpose, we denote the convex hull of a set  $A$  by  $\text{conv}(A)$  and put

$$\mathcal{W} = \{\varphi_p : p \in (0, 1)\},$$

where  $\varphi_p : [0, 1] \rightarrow [0, 1]$  is the function defined by (2).

**Proposition 5.1.** *The set  $\mathcal{W}$  is linearly independent. Moreover:*

- (i)  $\text{conv}(\mathcal{W}) \subset \mathcal{C}$ ;
- (ii)  $\text{conv}(\mathcal{W} \setminus \{\varphi_{\frac{1}{2}}\}) \subset \mathcal{C}_s$ .

*Proof.* To prove that  $\mathcal{W}$  is linearly independent fix  $n \in \mathbb{N}$ ,  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ ,  $0 < p_1 < p_2 < \dots < p_n < 1$  and assume that

$$\sum_{i=1}^n \alpha_i \varphi_{p_i}(x) = 0.$$

for every  $x \in [0, 1]$ . Applying (2) we conclude that  $\varphi_{p_i}(\frac{1}{2^k}) = p_i^k$  for all  $k \in \mathbb{N}$  and  $i \in \{1, \dots, n\}$ . Then for every  $k \in \mathbb{N}$  we have

$$\sum_{i=1}^n \alpha_i \left( \frac{p_i}{p_n} \right)^k = 0.$$

Taking the limit as  $k \rightarrow \infty$  we get  $\alpha_n = 0$ . Repeating this procedure  $n - 1$  times gives  $\alpha_n = \alpha_{n-1} = \dots = \alpha_1 = 0$ .

Assertion (i) follows from Remark 2.1 and assertion (ii) is a consequence of the moreover part of assertion (ii) of Theorem 1.1.  $\square$

To formulate an answer to the problem posed in [9] by Janusz Matkowski define first a function  $\varphi_1 : [0, 1] \rightarrow \mathbb{R}$  putting  $\varphi_1(x) = 1$  and observe that by Proposition 5.1 and the fact that  $\varphi_1(0) = 1$  and  $\varphi_p(0) = 0$  for every  $p \in (0, 1)$  the set  $\mathcal{W} \cup \{\varphi_1\}$  is linearly independent. Let  $\mathcal{M}$  denote the vector space whose basis is  $\mathcal{W} \cup \{\varphi_1\}$ , i.e.

$$\mathcal{M} = \text{lin}(\mathcal{W} \cup \{\varphi_1\}).$$

Applying Proposition 5.1 and Remark 1.2, we get the following result.

**Theorem 5.2.** *Every function belonging to  $\mathcal{M}$  is a continuous solution of equation (e<sub>1</sub>). Moreover,  $\sum_{i=1}^n \alpha_i \varphi_{p_i} \in \mathcal{M}$  is:*

- (i) *monotone provided that  $\text{sgn}(\alpha_i) = \text{sgn}(\alpha_j)$  for all  $i, j \in \{1, \dots, n\}$  such that  $p_i, p_j \in (0, 1)$ ;*

(ii) *singular for all  $p_1, \dots, p_n \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ .*

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